

A Noncommutative Residue on Tori and a Semiclassical Limit

Farzad Fathizadeh

Abstract

We define a noncommutative residue for classical Euclidean pseudodifferential operators on a torus of arbitrary dimension. We prove that, up to multiplication by a constant, it is the unique trace on the algebra of classical pseudodifferential operators modulo infinitely smoothing operators. In the case of the two torus, we show that the noncommutative residue is the semiclassical limit of a noncommutative residue defined on classical pseudodifferential operators on noncommutative two tori.

1 Introduction

Original works on noncommutative residues as traces on algebras of pseudodifferential operators can be traced back to [1, 6] of Adler and Manin on one dimensional symbols. In a remarkable work, Wodzicki defined the noncommutative residue in higher dimensions and proved that it is the unique trace on the algebra of pseudodifferential operators on compact manifolds [8]. The Wodzicki noncommutative residue has been generalized vastly in the context of the local index formula in noncommutative geometry by Connes and Moscovici [3]. In fact, using residue trace functionals and assuming the *simple discrete dimension spectrum hypothesis*, they show that the generalization of Wodzicki's residue is a trace on the algebra of pseudodifferential operators associated to a *spectral triple*.

In this paper we define a noncommutative residue for classical *Euclidean* pseudodifferential operators on tori [7] and prove that up to a constant multiple, it is the unique *continuous* trace on the algebra of classical pseudodifferential operators modulo infinitely smoothing operators. We also show that for the two torus, our noncommutative residue is the semiclassical limit of a noncommutative residue defined on classical pseudodifferential operators on noncommutative two tori [4].

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2 Euclidean Symbols on Tori

In this section we briefly recall a class of pseudodifferential operators on tori. We denote by $S^m(\mathbb{T}^n \times \mathbb{R}^n)$ the set of *Euclidean symbols* of order $m \in \mathbb{Z}$ on the torus $\mathbb{T}^n = \mathbb{R}^n/2\pi\mathbb{Z}^n$ [7]. This is the set of all C^∞ functions $\sigma : \mathbb{T}^n \times \mathbb{R}^n \rightarrow \mathbb{C}$ with the property that for all multi-indices α and β , there exists a positive constant $C_{\alpha,\beta}$ such that

$$|(\partial_x^\alpha \partial_\xi^\beta \sigma)(x, \xi)| \leq C_{\alpha,\beta} \langle \xi \rangle^{m-|\beta|}$$

for all $(x, \xi) \in \mathbb{T}^n \times \mathbb{R}^n$, where $\langle \xi \rangle = (1 + |\xi|^2)^{1/2}$. Throughout this paper, we assume that $n \geq 2$.

Let $\sigma \in S^m(\mathbb{T}^n \times \mathbb{R}^n)$. Then the corresponding *pseudodifferential operator* on \mathbb{T}^n is defined by

$$(T_\sigma f)(x) = \int_{\mathbb{R}^n} \int_{\mathbb{T}^n} e^{i(x-y) \cdot \xi} \sigma(x, \xi) f(y) dy d\xi, \quad x \in \mathbb{T}^n, \quad (2.1)$$

for all smooth functions f on \mathbb{T}^n . Pseudodifferential operators with symbols in $\cup_{m \in \mathbb{R}} S^m(\mathbb{T}^n \times \mathbb{R}^n)$ form an algebra [7]. This means that if $\sigma \in S^{m_1}(\mathbb{T}^n \times \mathbb{R}^n)$ and $\tau \in S^{m_2}(\mathbb{T}^n \times \mathbb{R}^n)$, then $T_\sigma T_\tau$ is a pseudodifferential operator of order $m_1 + m_2$ of which the symbol $\lambda \in S^{m_1+m_2}(\mathbb{T}^n \times \mathbb{R}^n)$ has an asymptotic expansion given by

$$\lambda \sim \sum_{\gamma} \frac{1}{\gamma!} (\partial_\xi^\gamma \sigma)(D_x^\gamma \tau). \quad (2.2)$$

A symbol $\sigma \in S^m(\mathbb{T}^n \times \mathbb{R}^n)$ is said to be *classical* if it admits an asymptotic expansion of the form

$$\sigma(x, \xi) \sim \sum_{j=0}^{\infty} \sigma_{m-j}(x, \xi) \quad \text{as } |\xi| \rightarrow \infty, \quad (2.3)$$

where each $\sigma_{m-j} : \mathbb{T}^n \times (\mathbb{R}^n \setminus \{0\}) \rightarrow \mathbb{C}$ is C^∞ and *positively homogeneous* in ξ of order $m-j$, i.e.,

$$\sigma_{m-j}(x, t\xi) = t^{m-j} \sigma_{m-j}(x, \xi)$$

for all $(x, \xi) \in \mathbb{T}^n \times (\mathbb{R}^n \setminus \{0\})$ and $t > 0$. The set of classical symbols of order m and the corresponding set of pseudodifferential operators are denoted by $S_{\text{cl}}^m(\mathbb{T}^n \times \mathbb{R}^n)$ and $\Psi_{\text{cl}}^m(\mathbb{T}^n \times \mathbb{R}^n)$ respectively. Using a similar argument to the one given in [4], we can see that the homogeneous terms in the above asymptotic expansion are uniquely determined by σ . The pseudodifferential operators associated with these symbols are also called *classical*.

3 A Noncommutative Residue

In this section we define a noncommutative residue for classical Euclidean pseudodifferential operators on tori and prove that up to a constant multiple, it

gives the unique *continuous* trace on the algebra of classical pseudodifferential operators. Let us denote the space of all classical Euclidean pseudodifferential operators with integral orders on \mathbb{T}^n by $\Psi_{\text{cl}}^\infty(\mathbb{T}^n \times \mathbb{R}^n)$. A linear functional $\varphi : \Psi_{\text{cl}}^\infty(\mathbb{T}^n \times \mathbb{R}^n) \rightarrow \mathbb{C}$ is said to be *continuous* if there exists an integer N such that φ vanishes on any pseudodifferential operator of order less than N .

Definition 3.1. Let $\sigma \in S_{\text{cl}}^m(\mathbb{T}^n \times \mathbb{R}^n)$ be such that it has an asymptotic expansion of the form

$$\sigma(x, \xi) \sim \sum_{j=0}^{\infty} \sigma_{m-j}(x, \xi) \text{ as } \xi \rightarrow \infty, \quad (3.1)$$

with positively homogeneous terms as in (2.3). Then the noncommutative residue of the pseudodifferential operator T_σ is defined by

$$\text{Res}(T_\sigma) = \int_{\mathbb{S}^{n-1}} \int_{\mathbb{T}^n} \sigma_{-n}(x, \xi) dx d\Omega, \quad (3.2)$$

where \mathbb{S}^{n-1} is the unit sphere in \mathbb{R}^n centered at the origin, and $d\Omega$ is the usual surface measure on the sphere.

Clearly, the noncommutative residue $\text{Res} : \Psi_{\text{cl}}^\infty(\mathbb{T}^n \times \mathbb{R}^n) \rightarrow \mathbb{C}$ is a linear functional that vanishes on any operator of order less than $-n$. In particular, it vanishes on the *infinitely smoothing operators* $\Psi_{\text{cl}}^{-\infty}(\mathbb{T}^n \times \mathbb{R}^n)$ defined by

$$\Psi_{\text{cl}}^{-\infty}(\mathbb{T}^n \times \mathbb{R}^n) = \cap_{m \in \mathbb{Z}} \Psi_{\text{cl}}^m(\mathbb{T}^n \times \mathbb{R}^n).$$

Theorem 3.2. The noncommutative residue $\text{Res} : \Psi_{\text{cl}}^\infty(\mathbb{T}^n \times \mathbb{R}^n) \rightarrow \mathbb{C}$ is a trace on the algebra of classical Euclidean pseudodifferential operators on \mathbb{T}^n . Moreover, it vanishes on the infinitely smoothing operators and up to multiplication by a constant, it is the unique continuous trace on this algebra.

Proof. Let $\sigma \in S_{\text{cl}}^m(\mathbb{T}^n \times \mathbb{R}^n)$ and let $\tau \in S_{\text{cl}}^{m'}(\mathbb{T}^n \times \mathbb{R}^n)$, where $m, m' \in \mathbb{Z}$. Suppose that σ and τ have asymptotic expansions given by

$$\sigma(x, \xi) \sim \sum_{j=0}^{\infty} \sigma_{m-j}(x, \xi)$$

and

$$\tau(x, \xi) \sim \sum_{k=0}^{\infty} \tau_{m'-k}(x, \xi)$$

as $|\xi| \rightarrow \infty$, where σ_{m-j} and $\tau_{m'-k}$ are homogeneous of order $m-j$ and $m'-k$ respectively. Using the product formula (2.2), the symbol λ of $T_\sigma T_\tau$ has the asymptotic expansion given by

$$\begin{aligned} \lambda &\sim \sum_{\gamma} \frac{1}{\gamma!} (\partial_\xi^\gamma \sigma)(D_x^\gamma \tau) \\ &\sim \sum_{\gamma} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{1}{\gamma!} (\partial_\xi^\gamma \sigma_{m-j})(D_x^\gamma \tau_{m'-k}). \end{aligned}$$

Since $\partial_\xi^\gamma \sigma_{m-j} D_x^\gamma \tau_{m'-k}$ is homogeneous of order $m - j - |\gamma| + m' - k$, by the definition of the noncommutative residue given by (3.2), we have

$$\begin{aligned} & \text{Res}(T_\sigma T_\tau) \\ &= \sum_{\substack{\gamma, j, k \geq 0 \\ m-j-|\gamma|+m'-k=-n}} \frac{1}{\gamma!} \int_{\mathbb{S}^{n-1}} \int_{\mathbb{T}^n} (\partial_\xi^\gamma \sigma_{m-j})(x, \xi) (D_x^\gamma \tau_{m'-k})(x, \xi) dx d\Omega. \end{aligned}$$

Similarly,

$$\begin{aligned} & \text{Res}(T_\tau T_\sigma) \\ &= \sum_{\substack{\gamma, k, j \geq 0 \\ m'-k-|\gamma|+m-j=-n}} \frac{1}{\gamma!} \int_{\mathbb{S}^{n-1}} \int_{\mathbb{T}^n} (\partial_\xi^\gamma \tau_{m'-k})(x, \xi) (D_x^\gamma \sigma_{m-j})(x, \xi) dx d\Omega. \end{aligned}$$

Therefore we have

$$\begin{aligned} & \text{Res}(T_\sigma T_\tau) - \text{Res}(T_\tau T_\sigma) \\ &= \sum \int_{\mathbb{S}^{n-1}} \int_{\mathbb{T}^n} \frac{1}{\gamma!} \{ (\partial_\xi^\gamma \sigma_{m-j})(x, \xi) (D_x^\gamma \tau_{m'-k})(x, \xi) - \\ & \quad (D_x^\gamma \sigma_{m-j})(x, \xi) (\partial_\xi^\gamma \tau_{m'-k})(x, \xi) \} dx d\Omega, \end{aligned}$$

where the summation is over all nonnegative integers j, k and multi-indices γ such that

$$m - j - |\gamma| + m' - k = -n.$$

Each term

$$(\partial_\xi^\gamma \sigma_{m-j})(x, \xi) (D_x^\gamma \tau_{m'-k})(x, \xi) - (D_x^\gamma \sigma_{m-j})(x, \xi) (\partial_\xi^\gamma \tau_{m'-k})(x, \xi)$$

in the above integral can be written in the form

$$\sum_{\ell=0}^n ((\partial_{\xi_\ell} A_\ell)(x, \xi) + (D_{x_\ell} B_\ell)(x, \xi)),$$

for some smooth maps A_ℓ and B_ℓ . Considering the order of homogeneity

$$m - j - \gamma + m' - k = -n,$$

each A_ℓ is homogeneous of order $-n + 1$, and each B_ℓ is homogeneous of order $-n$ in ξ . In view of Lemma 1.2 of [5], we have

$$\int_{\mathbb{S}^{n-1}} (\partial_{\xi_\ell} A_\ell)(x, \xi) d\Omega = 0,$$

and from integration by parts it follows that

$$\int_{\mathbb{T}^n} (D_{x_\ell} B_\ell)(x, \xi) dx = 0.$$

Hence we have proved that Res is a trace to the effect that

$$\text{Res}(T_\sigma T_\tau) - \text{Res}(T_\tau T_\sigma) = 0.$$

As for the uniqueness, let $\varphi : \Psi_{\text{cl}}^\infty(\mathbb{T}^n \times \mathbb{R}^n) / \Psi_{\text{cl}}^{-\infty}(\mathbb{T}^n \times \mathbb{R}^n) \rightarrow \mathbb{C}$ be a continuous trace on the algebra of classical pseudodifferential operators modulo the infinitely smoothing operators. For any classical symbol σ , the symbol of $T_{\xi_\ell} T_\sigma - T_\sigma T_{\xi_\ell}$ is equivalent to $D_{x_\ell} \sigma$. Therefore

$$\varphi(T_{D_{x_\ell}} \sigma) = 0, \quad \ell = 1, 2, \dots, n. \quad (3.3)$$

Also, φ vanishes on $T_\sigma T_{e^{ix_\ell}} - T_{e^{ix_\ell}} T_\sigma$, of which the symbol is equivalent to

$$(\partial_{\xi_\ell} \sigma) e^{ix_\ell} + \frac{1}{2!} (\partial_{\xi_\ell}^2 \sigma) e^{ix_\ell} + \frac{1}{3!} (\partial_{\xi_\ell}^3 \sigma) e^{ix_\ell} + \dots.$$

By iteration, namely, by using the same argument for $\partial_{\xi_\ell} \sigma$ instead of σ and so on, and using the continuity of φ , it follows that

$$\varphi(T_{\partial_{\xi_\ell}} \sigma) = 0, \quad \ell = 1, 2, \dots, n. \quad (3.4)$$

Now, assume that

$$\sigma(x, \xi) \sim \sum_{j=0}^{\infty} \sigma_{m-j}(x, \xi)$$

as $|\xi| \rightarrow \infty$, where each σ_{m-j} is homogeneous of order $m-j$. If $m-j \neq -n$, using Euler's identity, there are smooth map $h_{\ell, m-j}$ such that

$$\sigma_{m-j} = \sum_{\ell=0}^n \partial_{\xi_\ell} h_{\ell, m-j}.$$

Setting

$$h_\ell \sim \sum_{\substack{j \geq 0 \\ m-j \neq -n}} h_{\ell, m-j}, \quad \ell = 1, 2, \dots, n,$$

we have

$$\sigma \sim \sigma_{-n} + \sum_{\ell=1}^n \partial_{\xi_\ell} h_\ell.$$

Now, from (3.4) it follows that

$$\varphi(T_\sigma) = \varphi(T_{\sigma_{-n}}).$$

We write

$$\sigma_{-n}(x, \xi) = \sigma_{-n}(x, \xi) - r(x)|\xi|^{-n} + r(x)|\xi|^{-n},$$

where

$$r(x) = \frac{1}{|\mathbb{S}^{n-1}|} \int_{\mathbb{S}^{n-1}} \sigma_{-n}(x, \xi) \, d\Omega,$$

where $|\mathbb{S}^{n-1}|$ is the surface measure of \mathbb{S}^{n-1} . Since

$$\int_{\mathbb{S}^{n-1}} (\sigma_{-n}(x, \xi) - r(x)|\xi|^{-n}) d\Omega = 0,$$

it follows from Lemma 1.3 of [5] that $\sigma_{-n}(x, \xi) - r(x)|\xi|^{-n}$ can be written as a sum of partial derivatives. Hence, using (3.4), φ vanishes on the corresponding operator. Therefore

$$\varphi(T_\sigma) = \varphi(T_{r(x)|\xi|^{-n}}).$$

Now, we consider the linear map $\chi : C^\infty(\mathbb{T}^n) \rightarrow \mathbb{C}$ defined by

$$\chi(f) = \varphi(T_{f|\xi|^{-n}}).$$

Since

$$\chi(\partial_{x_\ell} f) = 0$$

for all $f \in C^\infty(\mathbb{T}^n)$ and $\ell = 1, 2, \dots, n$, there exists a constant c such that

$$\chi(f) = c \int_{\mathbb{T}^n} f(x) dx, \quad f \in C^\infty(\mathbb{T}^n).$$

Therefore

$$\begin{aligned} \varphi(T_\sigma) &= \int_{\mathbb{T}^n} r(x) dx \\ &= \frac{c}{|\mathbb{S}^{n-1}|} \int_{\mathbb{T}^n} \int_{\mathbb{S}^{n-1}} \sigma_{-n}(x, \xi) d\Omega dx \\ &= \frac{c}{|\mathbb{S}^{n-1}|} \text{Res}(T_\sigma). \end{aligned}$$

Therefore φ is a constant multiple of the noncommutative residue. \square

4 The Noncommutative Two Torus

In this section we first recall Connes' pseudodifferential calculus for the canonical dynamical system associated to the noncommutative two torus A_θ , $\theta \in \mathbb{R}$ [2]. Then we show that in the case $\theta = 0$, the noncommutative residue for classical pseudodifferential operators on A_θ defined in [4] coincides with the noncommutative residue defined in Section 3.

By definition, for a fixed $\theta \in \mathbb{R}$, A_θ is the universal unital C^* -algebra generated by two unitaries U and V satisfying

$$VU = e^{2\pi i \theta} UV.$$

There is a continuous action of \mathbb{T}^2 , $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$, on A_θ by C^* -algebra automorphisms $\{\alpha_s\}$, $s \in \mathbb{R}^2$, defined by

$$\alpha_s(U^m V^n) = e^{is \cdot (m, n)} U^m V^n. \quad (4.1)$$

The space of smooth elements for this action, *i.e.*, elements $a \in A_\theta$ for which the map $s \mapsto \alpha_s(a)$ is C^∞ , is denoted by A_θ^∞ . It is a dense subalgebra of A_θ which can be alternatively described as the algebra of elements in A_θ whose (noncommutative) Fourier expansion has rapidly decreasing coefficients. More precisely,

$$A_\theta^\infty = \left\{ \sum_{m,n \in \mathbb{Z}} a_{m,n} U^m V^n : \sup_{m,n \in \mathbb{Z}} (|m|^k |n|^q |a_{m,n}|) < \infty, k, q \in \mathbb{Z} \right\}.$$

There exist two derivations $\delta_1 : A_\theta^\infty \rightarrow A_\theta^\infty$ and $\delta_2 : A_\theta^\infty \rightarrow A_\theta^\infty$ corresponding to the above action of \mathbb{T}^2 on A_θ , which are, respectively, the analogs of the differential operators $-i \frac{\partial}{\partial x}$ and $-i \frac{\partial}{\partial y}$ on smooth functions on \mathbb{T}^2 . These derivations are fixed by

$$\delta_1(U) = U, \quad \delta_1(V) = 0,$$

and

$$\delta_2(U) = 0, \quad \delta_2(V) = V.$$

Moreover, for $j = 1, 2$, we have

$$\delta_j(a^*) = -\delta_j(a)^*$$

for all $a \in A_\theta^\infty$.

There is a normalized trace \mathfrak{t} on A_θ that turns out to be positive and faithful. This means that

$$\mathfrak{t}(1) = 1$$

and

$$\mathfrak{t}(a^* a) > 0$$

for all nonzero $a \in A_\theta$. The restriction of \mathfrak{t} to A_θ^∞ is given by

$$\mathfrak{t} \left(\sum_{j,k \in \mathbb{Z}} a_{j,k} U^j V^k \right) = a_{0,0}$$

for all

$$\sum_{j,k \in \mathbb{Z}} a_{j,k} U^j V^k \in A_\theta^\infty.$$

For any integer n , a smooth map $\sigma : \mathbb{R}^2 \rightarrow A_\theta^\infty$ is said to be a symbol of order n [2], if for all nonnegative integers i_1, i_2, j_1, j_2 , there exists a positive constant C , depending on i_1, i_2, j_1 and j_2 only, such that

$$||\delta_1^{i_1} \delta_2^{i_2} ((\partial_1^{j_1} \partial_2^{j_2} \sigma)(\xi))|| \leq c(1 + |\xi|)^{n-j_1-j_2},$$

and if there exists a smooth map $k : \mathbb{R}^2 \rightarrow A_\theta^\infty$ such that

$$\lim_{\lambda \rightarrow \infty} \lambda^{-n} \sigma(\lambda \xi_1, \lambda \xi_2) = k(\xi_1, \xi_2)$$

for all (ξ_1, ξ_2) in \mathbb{R}^2 . The space of symbols of order n is denoted by S^n .

To a symbol σ of order n , we associate an operator on A_θ^∞ [2], denoted by $T_{\sigma,\theta}$ and given by

$$T_{\sigma,\theta}(a) = (2\pi)^{-2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{-is \cdot \xi} \sigma(\xi) \alpha_s(a) \, ds \, d\xi, \quad a \in A_\theta^\infty, \quad (4.2)$$

where α_s is given by (4.1).

A symbol σ in S^n is said to be a *classical* symbol and we write $\sigma \in S_{cl}^n$ if it admits an asymptotic expansion of the form

$$\sigma(\xi) \sim \sum_{j=0}^{\infty} \sigma_{n-j}(\xi)$$

as $|\xi| \rightarrow \infty$, where for each $j = 0, 1, 2, \dots$, $\sigma_{n-j} : \mathbb{R}^2 \setminus \{0\} \rightarrow A_\theta^\infty$ is smooth and positively homogeneous of order $n-j$. Then we define the noncommutative residue $\text{Res}(T_{\sigma,\theta})$ of $T_{\sigma,\theta}$ by

$$\text{Res}(T_{\sigma,\theta}) = \int_{\mathbb{S}^1} \mathbf{t}(\sigma_{-2}(\xi)) \, d\Omega, \quad (4.3)$$

where $d\Omega$ is the Lebesgue measure on the unit circle \mathbb{S}^1 centered at the origin [4]. The space of pseudodifferential operators on A_θ form an algebra [2], and it is shown in [4] that the above noncommutative residue is the unique continuous trace on the classical operators.

Now, let us consider the noncommutative two torus A_θ for the case $\theta = 0$. By definition, we can assume that A_0 is the C^* -algebra generated by the functions U and V on \mathbb{R}^2 defined by

$$U(x, y) = e^{ix}$$

and

$$V(x, y) = e^{iy}$$

for all $(x, y) \in \mathbb{R}^2$. Since these functions are 2π -periodic in both variables, we can consider them as smooth functions defined on the two torus \mathbb{T}^2 , $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$. For all smooth function f defined on \mathbb{T}^2 , we use its Fourier expansion to obtain

$$f = \sum_{m,n \in \mathbb{Z}} a_{m,n} U^m V^n,$$

where

$$a_{m,n} = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} f(x, y) e^{-imx} e^{-iny} \, dx \, dy.$$

Hence in the case $\theta = 0$, A_θ is the algebra of continuous functions on the ordinary two torus and A_θ^∞ is the algebra of smooth functions on \mathbb{T}^2 .

So, if we translate the function f by $s = (s_1, s_2) \in \mathbb{R}^2$, and denote the result by $T_s f$, then

$$T_s f = \sum_{m,n \in \mathbb{Z}} e^{is \cdot (m,n)} a_{m,n} U^m V^n$$

because we have

$$\begin{aligned} & \int_0^{2\pi} \int_0^{2\pi} f(x + s_1, y + s_2) e^{-imx} e^{-iny} dx dy \\ &= e^{ims_1} e^{ins_2} \int_0^{2\pi} \int_0^{2\pi} f(x, y) e^{-imx} e^{-iny} dx dy. \end{aligned}$$

Thus, in the case $\theta = 0$, the action α_s on A_θ described in (4.1) is just the translation of functions by s . Now by a simple change of variable, namely by passing to $s = y - x$, we can observe the following identity for a variant of formula (2.1) for $n = 2$, and formula (4.2). In fact, we have

$$\begin{aligned} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{i(x-y)\cdot\xi} \sigma(x, \xi) f(y) dy d\xi &= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{-is\cdot\xi} \sigma(x, \xi) f(x + s) ds d\xi \\ &= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{-is\cdot\xi} \sigma(x, \xi) \alpha_s(f)(x) ds d\xi. \end{aligned}$$

Moreover, in the commutative case $\theta = 0$, the trace \mathfrak{t} on A_0 amounts to integration of continuous functions on the ordinary two torus \mathbb{T}^2 . We also note that in this case, the C^* -algebra norm is given by the supremum norm of continuous functions on \mathbb{T}^2 . Therefore the definition of the symbols given above coincide with the one given in Section 3. Note that here we are identifying the complex-valued smooth functions defined on $\mathbb{T}^2 \times \mathbb{R}^2$ with the smooth functions from \mathbb{R}^2 to A_θ^∞ .

Considering the above observations and the fact that the noncommutative residues (3.2) and (4.3) are defined on algebras of classical pseudodifferential symbols with multiplications induced from composition of pseudodifferential operators, it is clear that for $n = 2$, the noncommutative residue (3.2) is the semiclassical limit of (4.3) defined on the pseudodifferential symbols on the noncommutative two torus A_θ . We record this result in the following.

Theorem 4.1. *In the case $\theta = 0$, the noncommutative residue defined by (4.3) on classical pseudodifferential operators on the noncommutative two torus A_θ , coincides with the noncommutative residue defined by (3.2) on the classical Euclidean pseudodifferential operators on \mathbb{T}^2 .*

References

- [1] M. Adler, On a trace functional for formal pseudo-differential operators and the symplectic structure of Korteweg–de Vries type equations, *Invent. Math.* **50** (1978/79), 219–248.
- [2] A. Connes, C^* -algèbres et géométrie différentielle, *C.R. Acad. Sc. Paris Sér. A-B* **290** (1980), A599–A604.
- [3] A. Connes and H. Moscovici, The local index formula in noncommutative geometry. *Geom. Funct. Anal.* **5**, no. 2 (1995), 174–243.

- [4] F. Fathizadeh and M. W. Wong, Noncommutative residues for pseudo-differential operators on the noncommutative two-torus, *Journal of Pseudo-Differential Operators and Applications*, **2**(3) (2011), 289–302.
- [5] B. Fedosov, F. Golse, E. Leichtnam and E. Schrohe, The noncommutative residue for manifolds with boundary, *J. Funct. Anal.* **142**(1) (1996), 1–31.
- [6] Ju. I. Manin, Algebraic aspects of nonlinear differential equations, *J. Soviet Math.* **11** (1979), 1–122.
- [7] M. Ruzhansky and V. Turunen, Pseudo-Differential Operators and Symmetries, Birkhäuser, 2010.
- [8] M. Wodzicki, Noncommutative residue. I. Fundamentals, in *K-theory, Arithmetic and Geometry (Moscow, 1984–1986)*, Lecture Notes in Mathematics **1289**, Springer, 1987, 320–399.

Department of Mathematics and Statistics
 York University
 Toronto, Ontario, Canada, M3J 1P3
 ffathiza@mathstat.yorku.ca